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On the Stochastic Maximum Principle in Banach Space

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1. INTRODUCTION

We initiate the study of stochastic control theory in infinite dimensional Banach space by proving an infinite dimensional analog of the celebrated Pontryagin's maximum principle [12] and its stochastic version due to Kushner [11]. We prove also an existence theorem for optimal controls. For the finite dimensional theory, see [1–5, 13] and the references in the comprehensive review paper [4]. Our approach in proving the maximum principle is strongly influenced by [13] where the stochastic integral is McShane's belated integral. However, we use a new machinery introduced in [10]. Also, our existence theorem is new even if the Banach space is finite dimensional.

It is well-known that a real separable Banach space can be regarded as an abstract Wiener space [6]. Thus we will work on stochastic systems in a fixed abstract Wiener space $H \subset B$. Consider the following stochastic integral equation

$$X_t = x + \int_0^t A(s, X_s)\, dW_s + \int_0^t \sigma(s, X_s, u(s))\, ds, \qquad 0 \leqslant t \leqslant \tau, \tag{1}$$

where τ is a fixed time, W_t is a Wiener process in $H \subset B$ and the control function u takes values in a subset U of a separable Banach space G (finite or infinite dimensional). Similar type of Eq. (1) has been studied in [8]. Let

$$C(u, X) = \int_0^\tau f(t, X_t, u(t))\, dt + p(X_\tau), \tag{2}$$

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and

$$\Phi(u, X) = E[C(u, X)] \quad \text{(the expectation).} \tag{3}$$

The regularity assumptions on the functions A, σ, f and p will be made precise in Section 2. Our object is to seek a necessary condition for u_0 such that u_0 and the corresponding trajectory X_0 will minimize Φ over the set of all u and X satisfying Eq. (1). This necessary condition is given in Theorem 1 (Section 2). As in the finite dimensional case it involves the Lagrange multiplier ϕ_t. However, a dissimilarity occurs in infinite dimensional case, namely, ϕ_t is a stochastic process satisfying a linear integral equation in the Hilbert space H (instead of B). Section 3 is devoted to the proof of Theorem 1 and in Section 4 we give two examples, one in which B is infinite dimensional and another B is one-dimensional.

Next, consider the control system (1)–(2) with the diffusion coefficients and the cost function given as follows:

$$\left.\begin{aligned} A(t, x) = J + K(t, x), \qquad \sigma(t, x, u) = C(t)x + \theta(t, u),\\ f(t, x, u) = | Dx |^2 + \langle F(t)x, \zeta(t, u)\rangle + g(t, u), \qquad p = 0.\end{aligned}\right\} \tag{4}$$

Under a rather mild assumption on (4), we have, in Theorem 2 (Section 2), the existence of an optimal control for Φ if we restrict Φ to a certain class of control functions. In fact, this restriction is motivated by Theorem 2.3 of [1]. The proof of Theorem 2 is given in Section 5.

2. Maximum Principle and Existence Theorem

Notation: $| \cdot | = H$-norm, $\| \cdot \| = B$-norm, $| \cdot |_0 = G$-norm, $\langle\ ,\ \rangle = H$-inner product, $(\ ,\ )$ = natural pairing of B^* and B, ($B^* \subset H \subset B$ as in [8]).

The appropriate smoothness for functions defined in the abstract Wiener space $H \subset B$ is the so-called Fréchet differentiability in H-directions (shortly, H-differentiability). A function f from B into another Banach space Z is said to be H-differentiable at x if there exists a (unique) linear operator $T \in L(H, Z)$ ($\equiv$ the Banach space of linear operators from H into Z with operator norm $\| \cdot \|_{H,Z}$) such that $\|f(x + h) - f(x) - Th\|_Z = o(| h |)$, $h \in H$. T will be denoted by $f'(x)$ or f_x (specially, when f depends also on other variables). f is said to be C_H^1 if $f'(x)$ exists for all x in B and f' is continuous from B into $L(H, Z)$.

We now state the hypotheses of A, σ, f and p in Eqs. (1) and (2).

(A-1). $A(t, x) = J + K(t, x)$, where $J \in L(B, B)$ and K is a continuous map from $[0, \tau] \times B$ into $L_2(H)$ ($\equiv$ the Hilbert space of Schmidt operators of H with Schmidt norm $\|\cdot\|_2$).

(A-2). There is a constant c such that $\| K(t, x) - K(t, y)\|_2 \leqslant c \| x - y \|$, $\| K(t, x)\|_2 \leqslant c(1 + \| x \|)$ for all t and x.

(A-3). K is $C_H{}^1$ in x variable such that K_x is bounded and continuous from $[0, \tau] \times B$ into $L(H, L_2(H))$.

(σ-1). σ is a continuous map from $[0, \tau] \times B \times U$ into H such that $| \sigma(t, x, u) - \sigma(t, y, v)| \leqslant c(\| x - y \| + | u - v |_0)$ and $| \sigma(t, x, u)| \leqslant c(1 + \| x \| + | u |_0)$.

(σ-2). σ is $C_H{}^1$ in x variable such that σ_x is bounded and continuous from $[0, \tau] \times B \times U$ into $L(H, H)$.

(f-1). f is a real-valued continuous function in $[0, \tau] \times B \times U$ such that $| f(t, x, u)| \leqslant c(1 + \| x \|^2 + | u |_0^2)$.

(f-2). f is $C_H{}^1$ in x variable such that $| f_x(t, x, u)| \leqslant c(1 + \|x\| + | u |_0^2)$.

(p-1). p is a real-valued $C_H{}^1$ continuous function in B such that $| p'(x)| \leqslant c(1 + \| x \|)$.

We now specify the control functions space $\mathscr{U}$. $\mathscr{U}$ consists of all square integrable functions $u(t)$ in $[0, \tau]$ taking values in U, i.e., $\| u \|_0^2 = \int_0^\tau | u(t)|_0^2 \, dt < \infty$.

The adjoint system of (1)–(2) is defined to be the following linear stochastic differential equation with state space the Hilbert space H,

$$
\begin{aligned}
d\phi_t &= -A_x(t, X_t)^* \phi_t \, dW_t - [\sigma_x(t, X_t, u(t))^* \phi_t + f_x(t, X_t, u(t))] \, dt \\
\phi_\tau &= p'(X_\tau),
\end{aligned}
\tag{5}
$$

where if $T \in L(H, H)$ then T^* denotes the adjoint operator of T, and if $S \in L(H, L(H, H))$ then $S^* \in L(H, L(H, H))$ is defined as follows: $S^*(h) = [S(h)]^*$, $h \in H$. The Hamiltonian $\mathscr{H}$ is defined for each t in $[0, \tau]$ to be a function in $H \times B \times U$ as follows:

$$
\mathscr{H}(\phi, x, u; t) = f(t, x, u) + \langle \phi, \sigma(t, x, u) \rangle. \tag{6}
$$

Note that (A-1), (A-2),..., (p-1) imply that both Eqs. (1) and (5) have unique continuous nonanticipating solutions for each $u(\cdot) \in \mathscr{U}$ and $E \int_0^\tau \| X_t \|^2 \, dt < \infty$, $E \int_0^\tau | \phi_t |^2 \, dt < \infty$. This follows directly from the existence theorems in [8] and [10].

THEOREM 1 (Stochastic Maximum Principle). *Assume the hypotheses* $(A\text{-}1), (A\text{-}2),..., (p\text{-}1)$. *Let* $u(\cdot) \in \mathscr{U}$ *and* X_t *be the corresponding trajectory (i.e., solution of* (1)) *and* ϕ_t *be the corresponding Lagrange multiplier (i.e., solution of* (5)). *Then in order for* $u(\cdot)$ *to minimize* Φ *in Eq.* (3) *over* $\mathscr{U}$ *it is necessary that for almost all t in* $[0, \tau]$,

$$\inf_{u \in U} E[\mathscr{H}(\phi_t, X_t, u; t)] = E[\mathscr{H}(\phi_t, X_t, u(t); t)]. \tag{7}$$

We now consider the existence problem. In this case we suppose G is a Hilbert space with inner product $\langle\ ,\ \rangle_0$ and U a closed subspace. Then $\mathscr{U}$ is also a Hilbert space with inner product $\langle\langle u(\cdot), v(\cdot)\rangle\rangle_0 = \int_0^\tau \langle u(t), v(t)\rangle_0 \, dt$. $\mathscr{U}$ is obviously separable. Let $k(s, t)$ be a symmetric square integrable kernel function in $[0, \tau] \times [0, \tau]$, i.e., $\int_0^\tau \int_0^\tau | k(s, t)|^2 \, ds \, dt < \infty$. Define an integral operator T from $\mathscr{U}$ into itself by

$$Tu(s) = \int_0^\tau k(s, t) \, u(t) \, dt, \qquad 0 \leqslant s \leqslant \tau. \tag{8}$$

Let $\mathscr{U}_{T,r}$ $(r > 0)$ consist of all control functions of the form Tu, where $\| u \|_0 \leqslant r$.

THEOREM 2 (Existence for Optimal Controls). *In* (4) *suppose*: (i) A *satisfies* $(A\text{-}1)$ *and* $(A\text{-}2)$, (ii) C *and* F *are bounded continuous maps from* $[0, \tau]$ *into* $L(B, H)$, *and* $D \in L(B, H)$, (iii) θ *and* ζ *are bounded continuous maps from* $[0, \tau] \times U$ *into* H, *and* (iv) g *is a real-valued bounded continuous function on* $[0, \tau] \times U$. *Assume* $DK(t, x) = 0$ *for all t and x and* $DC(t) = \alpha(t)D$ *for some real-valued bounded continuous function* α. *Furthermore, suppose* $X_0 = 0$ *in Eq.* (1). *Then the functional* Φ *in Eq.* (3) *has both maximum and minimum over* $\mathscr{U}_{T,r}$.

Remark. $X_0 = 0$ is only a technical assumption for the proof.

3. PROOF OF THEOREM 1

We state a lemma which can be proved in the same way as Lemma A.2 [10].

LEMMA 1. *Suppose* X_t *is the solution of Eq.* (1) *corresponding to* $u(\cdot) \in \mathscr{U}$. *Then, for* $0 \leqslant t, s \leqslant \tau$,

$$E \| X_s - X_t \|^2 \leqslant c_1(e^{c_1|s-t|} - 1)(1 + E \| X_t \|^2 + \| u \|_0^2),$$

where c_1 is a constant depending only on the Lipschitzian constant c in the hypothesis and the norm $\| J \|_{B,B}$ of J.

The next lemma deals with the variation of controls. Let $u(\cdot) \in \mathscr{U}$ and X_t be the corresponding trajectory. Let t_0 be a Lebesgue point of $u(\cdot)$ (see, e.g., [7]) and $\epsilon > 0$, $z \in U$, define the perturbed control function $u_{\epsilon,z}(\cdot)$ of $u(\cdot)$ as follows:

$$u_{\epsilon,z}(t) = \begin{cases} u(t) & 0 \leqslant t \leqslant t_0 - \epsilon \\ z & t_0 - \epsilon < t \leqslant t_0 \\ u(t) & t_0 < t \leqslant \tau. \end{cases}$$

Note that $u_{\epsilon,z}(\cdot) \in \mathscr{U}$ and let $X_t^{\epsilon,z}$ be the corresponding trajectory such that $X_t^{\epsilon,z} = X_0$. We point out that $X_t^{\epsilon,z} - X_t$ is a stochastic process in H (rather than B). This will become clear through the following lemmas and the proofs.

Lemma 2. *Let t_0 be a Lebesgue point of $u(\cdot)$ then*

$$E \mid X_{t_0}^{\epsilon,z} - X_{t_0} - \epsilon[\sigma(t_0, X_{t_0}, z) - \sigma(t_0, X_{t_0}, u(t_0))] \mid^2 = o(\epsilon^2).$$

Proof. By the uniqueness of solution it follows that $X_{t_0-\epsilon}^{\epsilon,z} = X_{t_0-\epsilon}$ since $X_0^{\epsilon,z} = X_0$. In order to simplify notation, set $Y_t = X_t^{\epsilon,z}$, $t_0 - \epsilon < t \leqslant t_0$. Then

$$Y_t = X_{t_0-\epsilon} + \int_{t_0-\epsilon}^{t} A(s, Y_s)\, dW_s + \int_{t_0-\epsilon}^{t} \sigma(s, Y_s, z)\, ds$$

and

$$X_t = X_{t_0-\epsilon} + \int_{t_0-\epsilon}^{t} A(s, X_s)\, dW_s + \int_{t_0-\epsilon}^{t} \sigma(s, X_s, u(s))\, ds.$$

Let

$$\lambda(t) = Y_t - X_t - (t - t_0 + \epsilon)[\sigma(t_0, X_{t_0}, z) - \sigma(t_0, X_{t_0}, u(t_0))]. \tag{10}$$

It can be checked easily that

$$\lambda(t) = \mathrm{I} + \mathrm{II} - \mathrm{III}, \tag{11}$$

where

$$\mathrm{I} = \int_{t_0-\epsilon}^{t} [A(s, Y_s) - A(s, X_s)]\, dW_s,$$

$$\mathrm{II} = \int_{t_0-\epsilon}^{t} [\sigma(s, Y_s, z) - \sigma(t_0, X_{t_0}, z)]\, ds,$$

and

$$\mathrm{III} = \int_{t_0-\epsilon}^{t} [\sigma(s, X_s, u(s)) - \sigma(t_0, X_{t_0}, u(t_0))]\, ds.$$

In the following let c_2 denote a general constant depending only on c (Lipschitzian constant in the hypotheses) and c_1 (in Lemma 1).

Clearly,

$$\mathrm{I} = \int_{t_0-\epsilon}^{t} [K(s, Y_s) - K(s, X_s)]\, dW_s,$$

and by Proposition 3.1 in [8],

$$E\,|\,\mathrm{I}\,|^2 = E\int_{t_0-\epsilon}^{t} \|\, K(s, Y_s) - K(s, X_s)\|_2^2\, ds.$$

$$\leqslant c_2 E \int_{t_0-\epsilon}^{t} \|\, Y_s - X_s\,\|^2\, ds.$$

Let

$$\beta(t) = Y_t - X_t \qquad t_0 - \epsilon < t \leqslant t_0 \tag{12}$$

then

$$E\,|\,\mathrm{I}\,|^2 \leqslant c_2 \int_{t_0-\epsilon}^{t} E\,\|\,\beta(s)\|^2\, ds. \tag{13}$$

Writing $\sigma(s, Y_s, z) - \sigma(t_0, X_{t_0}, z)$ as $[\sigma(s, Y_s, z) - \sigma(s, X_{t_0}, z)] + [\sigma(s, X_{t_0}, z) - \sigma(t_0, X_{t_0}, z)]$ and using (σ-1), we can estimate II as follows:

$$E\,|\,\mathrm{II}\,|^2 \leqslant h(t) + c_2 \int_{t_0-\epsilon}^{t} E\,\|\,\beta(s)\|^2\, ds, \tag{14}$$

where

$$h(t) = c_2\epsilon \int_{t_0-\epsilon}^{t} [E\,\|\,X_s - X_{t_0}\|^2 + E\,|\,\sigma(s, X_{t_0}, z) - \sigma(t_0, X_{t_0}, z)|^2]\, ds.$$

Similarly, we have

$$E\,|\,\mathrm{III}\,|^3 \leqslant k(t) + c_2 \int_{t_0-\epsilon}^{t} E\,\|\,\beta(s)\|^2\, ds, \tag{15}$$

where

$$k(t) = c_2\epsilon \int_{t_0-\epsilon}^{t} [E\,\|\,X_s - X_{t_0}\|^2 + E\,|\,\sigma(s, X_{t_0}, u(t_0)) - \sigma(t_0, X_{t_0}, u(t_0))|^2$$
$$+ |\,u(s) - u(t_0)|_0^2]\, ds.$$

Putting (11), (13), (14), and (15) together, we obtain

$$E\,|\,\lambda(t)|^2 \leqslant h(t) + k(t) + c_2 \int_{t_0-\epsilon}^{t} E\,\|\,\beta(s)\|^2\,ds. \tag{16}$$

On the other hand, from (1) we get

$$Y_t - X_t = \lambda(t) + (t - t_0 + \epsilon)[\sigma(t_0\,,X_{t_0}\,,z) - \sigma(t_0\,,X_{t_0}\,,u(t_0)],$$

whence

$$\begin{aligned} E\,\|\,\beta(t)\|^2 &\leqslant c_2 E\,\|\,\lambda(t)\|^2 + c_2\epsilon^2 E\,\|\,\sigma(t_0\,,X_{t_0}\,,z) - \sigma(t_0\,,X_{t_0}\,,u(t_0))\|^2 \\ &\leqslant c_2 E\,|\,\lambda(t)|^2 + c_2\epsilon^2 E\,|\,\sigma(t_0\,,X_{t_0}\,,z) - \sigma(t_0\,,X_{t_0}\,,u(t_0))|^2 \\ &\leqslant c_2 \int_{t_0-\epsilon}^{t} E\,\|\,\beta(s)\|^2\,ds + h(t) + k(t) + c_2\epsilon^2\,|\,z - u(t_0)|_0^2 \\ &\qquad\qquad\qquad\qquad \text{by (16) and } (\sigma\text{-}1). \end{aligned}$$

Thus by Gronwall's inequality,

$$\begin{aligned} E\,\|\,\beta(t)\|^2 &\leqslant h(t) + k(t) + c_2\epsilon^2\,|\,z - u(t_0)|_0^2 \\ &\quad + c_2 \int_{t_0-\epsilon}^{t} e^{c_2(t-s)}\,[h(s) + k(s) + c_2\epsilon^2\,|\,z - u(t_0)|_0^2]\,ds. \end{aligned}$$

Note that $h(t)$ and $k(t)$ are increasing functions of t, hence from the above inequality we have

$$E\,\|\,\beta(t)\|^2 \leqslant c_2[h(t_0) + k(t_0) + \epsilon^2\,|\,z - u(t_0)|_0^2]. \tag{17}$$

From (16) and (17) it is easy to see that

$$E\,|\,\lambda(t_0)|^2 \leqslant c_2[h(t_0) + k(t_0) + \epsilon^3\,|\,z - u(t_0)|_0^2].$$

To finish the proof, simply note that both $h(t_0)$ and $k(t_0)$ are $o(\epsilon^2)$ by Lemma 1 and the fact that $E\,|\,\sigma(s, X_{t_0}\,, z) - \sigma(t_0\,, X_{t_0}\,, z)|^2$ and $E\,|\,\sigma(s, X_{t_0}\,, u(t_0)) - \sigma(t_0\,, X_{t_0}\,, u(t_0))|^2$ are continuous functions of s, and also, note that $\int_{t_0-\epsilon}^{t_0} |\,u(s) - u(t_0)|_0^2\,ds = o(\epsilon)$ since t_0 is a Lebesgue point of $u(\cdot)$.

LEMMA 3. *If t_0 is a Lebesgue point of $u(\cdot)$ and $E[f(\cdot\,, X.\,, u(\cdot))]$ then*

$$\begin{aligned} \lim_{\epsilon\downarrow 0} \epsilon^{-1} E \int_{t_0-\epsilon}^{t_0} [f(t, X_t^{\epsilon,z}, z) - f(t, X_t\,, u(t))]\,dt \\ = E[f(t_0\,, X_{t_0}\,, z) - f(t_0\,, X_{t_0}\,, u(t_0))]. \end{aligned}$$

Proof. Let

$$\xi(t) = E[f(t, X_t, u(t))], \qquad 0 \leqslant t \leqslant \tau.$$

It follows from (f-1) that

$$|\xi(t)| \leqslant c(1 + E\|X_t\|^2 + |u(t)|_0^2).$$

Lemma 1 gives $\int_0^\tau E\|X_t\|^2\,dt < \infty$, hence $\int_0^\tau |\xi(t)|\,dt < \infty$. Therefore,

$$\lim_{\epsilon\downarrow 0} \epsilon^{-1} E \int_{t_0-\epsilon}^{t_0} f(t, X_t, u(t))\,dt = E[f(t_0, X_{t_0}, u(t_0))]. \tag{18}$$

Similarly,

$$\lim_{\epsilon\downarrow 0} \epsilon^{-1} E \int_{t_0-\epsilon}^{t_0} f(t, X_t, z)\,dt = E[f(t_0, X_{t_0}, z)]. \tag{19}$$

On the other hand, since f is $C_H{}^1$ in x variable,

$$f(t, X_t^{\epsilon,z}, z) - f(t, X_t, z) = \langle f_x(t\ X_t, z), X_t^{\epsilon,z} - X_t\rangle + o(|X_t^{\epsilon,z} - X_t|).$$

Use (f-2) and apply Lemma 2 to conclude that

$$\lim_{\epsilon\downarrow 0} \epsilon^{-1} \int_{t_0-\epsilon}^{t_0} E\,|f(t, X_t^{\epsilon,z}, z) - f(t, X_t, z)|^2\,dt = 0. \tag{20}$$

Equations (18), (19), and (20) give the assertion of the lemma.

In the interval $t_0 < t \leqslant \tau$, X_t and $X_t^{\epsilon,z}$ satisfy the same stochastic differential equation (with initial distributions X_{t_0} and $X_{t_0}^{\epsilon,z}$ at t_0, respectively), namely,

$$dV_t = A(t, V_t)\,dW_t + \sigma(t, V_t, u(t))\,dt. \tag{21}$$

Now we will employ some results in [10] to study Eq. (21). Let us recall the notion of mean-square differentiability in H-directions (MS-H-differentiability) introduced in [10, Definition 5]. With a slight modification of Theorem 7 [10] in order to apply to our case here we conclude that the solution of Eq. (21) is MS-H-differentiable. The MS-H-derivative of V_t at X_{t_0} is given by the solution of the following $L(H, H)$-valued stochastic equation, $t_0 \leqslant t \leqslant \tau$,

$$Y_{t_0,t} = I + \int_{t_0}^{t} Y_{t_0,s}\,\Delta K_x(s, X_s)\,dW_s + \int_{t_0}^{t} Y_{t_0,s} \circ \sigma_x(s, X_s, u(s))\,ds, \tag{22}$$

where I is the identity operator of H.

Remark. For $S \in L(H, L(H, H))$ and $T \in L(H, H)$, $T \varDelta S \in L(H, L(H, H))$ is defined by: $T \varDelta S(h) = T \circ [S(h)]$, $h \in H$, where $\circ$ is the composition of two operators. We define also that $S \varDelta T(h) = S(h) \circ T$. For a detailed discussion of the above operator-valued stochastic integral equation, see [9 and 10].

LEMMA 4. *Let t_0 be a Lebesgue point of $u(\cdot)$ then*

$$\lim_{\epsilon \downarrow 0} \epsilon^{-1} E \int_{t_0}^{\tau} [f(t, X_t^{\epsilon,z}, u(t)) - f(t, X_t, u(t))] \, dt$$

$$= E \int_{t_0}^{\tau} \langle f_x(t, X_t, u(t)), Y_{t_0,t}(\sigma(t_0, X_{t_0}, z) - \sigma(t_0, X_{t_0}, u(t_0)))\rangle \, dt,$$

where $Y_{t_0,t}$ is the solution of Eq. (22).

Proof. First note that using the same arguement in the proof of Lemma 2 we see that $E \mid X_t^{\epsilon,z} - X_t \mid^2$ is dominated by $E \mid X_{t_0}^{\epsilon,z} - X_{t_0} \mid^2$ which is $o(\epsilon^2)$ by Lemma 2. Hence $E \mid X_t^{\epsilon,z} - X_t \mid$ is $o(\epsilon)$ uniformly for $t_0 < t \leqslant \tau$. Therefore,

$$\lim_{\epsilon \downarrow 0} \epsilon^{-1} E \int_{t_0}^{\tau} [f(t, X_t^{\epsilon,z}, u(t)) - f(t, X_t, u(t))] \, dt$$

$$= \lim_{\epsilon \downarrow 0} \epsilon^{-1} E \int_{t_0}^{\tau} \langle f_x(t, X_t, u(t)), X_t^{\epsilon,z} - X_t\rangle \, dt. \tag{23}$$

Let $Y_{t_0,t,x}$ denote the MS-H-derivative of V_t at x then

$$X_t^{\epsilon,z} - X_t = \int_0^1 Y_{t_0,t,X_{t_0}+\lambda(X_{t_0}^{\epsilon,z}-X_{t_0})}(X_{t_0}^{\epsilon,z} - X_{t_0}) \, d\lambda, \qquad t_0 < t \leqslant \tau.$$

Using (A-3) and (σ-2) to show

$$E \| Y_{t_0,t,X_{t_0}+\lambda(X_{t_0}^{\epsilon,z}-X_{t_0})} \|_{H,H}$$

is uniformly bounded for $\epsilon > 0$ and then Lebesgue dominated convergence theorem and Lemma 2 to find the limit, we conclude that almost surely

$$\lim_{\epsilon \downarrow 0} \epsilon^{-1}(X_t^{\epsilon,z} - X_t) = Y_{t_0,t,X_{t_0}}(\sigma(t_0, X_{t_0}, z) - \sigma(t_0, X_{t_0}, u(t_0))),$$

which together with (23) yield the lemma. Note that $Y_{t_0,t,X_{t_0}}$ is the solution of Eq. (22).

LEMMA 5.

$$\lim_{\epsilon\downarrow 0} \epsilon^{-1}E[p(X_\tau^{\epsilon,z}) - p(X_\tau)]$$
$$= E\langle p'(X_\tau), Y_{t_0,\tau}(\sigma(t_0, X_{t_0}, z) - \sigma(t_0, X_{t_0}, u(t_0)))\rangle,$$

where $Y_{t_0,t}$ is the same as Lemma 4, i.e., the solution of Eq. (22).

Proof. Similar to the previous one.

From now on, we assume $u(\cdot)$ is an optimal control minimizing Φ in Eq. (3) and X_t the corresponding optimal trajectory. Define

$$\Lambda_{z,t_0}(\epsilon) = \begin{cases} \Phi(u_{\epsilon,z}, X^{\epsilon,z}), & \epsilon > 0 \\ \Phi(u, X), & \epsilon = 0 \end{cases} \tag{24}$$

where $z \in U$ and $u_{\epsilon,z}$ is given in (9) and $X^{\epsilon,z}$ is the corresponding trajectory with $X_0^{\epsilon,z} = X_0$.

PROPOSITION 1. *If t_0 is a Lebesgue point of $u(\cdot)$ and $E[f(\cdot, X_\cdot, u(\cdot))]$ then the right derivative $\Lambda'_{z,t_0}(0)$ of Λ_{z,t_0} at the origin exists. Furthermore, it is given by:*

$$\begin{aligned}\Lambda'_{z,t_0}(0) &= E[f(t_0, X_{t_0}, z) - f(t_0, X_{t_0}, u(t_0))] \\ &\quad + E\int_{t_0}^{\tau} \langle f_x(t, X_t, u(t)), Y_{t_0,t}(\sigma(t_0, X_{t_0}, z) - \sigma(t_0, X_{t_0}, u(t_0)))\rangle\, dt \\ &\quad + E\langle p'(X_\tau), Y_{t_0,\tau}(\sigma(t_0, X_{t_0}, z) - \sigma(t_0, X_{t_0}, u(t_0)))\rangle, \end{aligned} \tag{25}$$

where $Y_{t_0,t} (t_0 \leqslant t \leqslant \tau)$ is the solution of Eq. (22).

Proof. Note that in the interval $0 \leqslant t \leqslant t_0 - \epsilon$, $u_{\epsilon,z}(t) = u(t)$. Hence by the uniqueness, $X_t^{\epsilon,z} = X_t$. Hence

$$E\int_0^{t_0-\epsilon} [f(t, X_t^{\epsilon,z}, u_{\epsilon,z}(t)) - f(t, X_t, u(t))]\, dt = 0.$$

This rest follows immediately from Lemmas 3, 4, and 5.

Define a stochastic process with state space H by:

$$\phi_s = \int_s^{\tau} Y^*_{s,t}(f_x(t, X_t, u(t)))\, dt + Y^*_{s,\tau}(p'(X_\tau)), \qquad 0 \leqslant s \leqslant \tau. \tag{26}$$

Use this ϕ_s and the Hamiltonian $\mathscr{H}$ defined in (6), (25) can then be rewritten as,

$$\Lambda'_{z,t_0}(0) = E\mathscr{H}(\phi_{t_0}, X_{t_0}, z; t_0) - E\mathscr{H}(\phi_{t_0}, X_{t_0}, u(t_0); t_0).$$

Now, in order for $u(\cdot)$ to minimize Φ, it is necessary to have $\Lambda'_{z,t_0}(0) \geqslant 0$, i.e.,

$$E\mathscr{H}(\phi_{t_0}, X_{t_0}, z; t_0) \geqslant E\mathscr{H}(\phi_{t_0}, X_{t_0}, u(t_0); t_0)$$

for all $z \in U$ and all the Lebesgue points t_0 of $u(\cdot)$ and $E[f(\cdot, X., u(\cdot))]$. Clearly, for all those t_0,

$$\inf_{z\in U} E\mathscr{H}(\phi_{t_0}, X_{t_0}, z; t_0) = E\mathscr{H}(\phi_{t_0}, X_{t_0}, u(t_0); t_0).$$

Of course, it is well-known that those t_0's are almost everywhere in $[0, \tau]$. Thus we will finish the proof if we show that ϕ_s defined in (26) satisfies Eq. (5). First note that from Eq. (22) $Y_{\tau,\tau} = I$, the identity operator of H. Hence $\phi_\tau = p'(X_\tau)$. From (26), we derive easily the stochastic differential of ϕ_s,

$$\begin{aligned} d\phi_s = {} & -Y^*_{s,s}(f_x(s\, X_s, , u(s)))\, ds + \int_s^\tau (d_s Y^*_{s,t})(f_x(t, X_t, u(t)))\, dt \\ & + (d_s Y^*_{s,\tau})(p'(X_\tau)), \end{aligned} \tag{27}$$

where d_s indicates the differentials with respect to s-variable. On the other hand, from Eq. (22) we have

$$Y_{s,t} = I + \int_s^t Y_{s,\lambda}\, \Delta K_x(\lambda, X_\lambda)\, dW_\lambda + \int_s^t Y_{s,\lambda} \circ \sigma_x(\lambda, X_\lambda, u(\lambda))\, ds,$$

and, taking adjoint, we have

$$Y^*_{s,t} = I + \int_s^t K_x(\lambda, X_\lambda)^*\, \Delta Y^*_{s,\lambda}\, dW_\lambda + \int_s^t \sigma_x(\lambda, X_\lambda, u(\lambda))^* \circ Y^*_{s,\lambda}\, d\lambda,$$

where if $S \in L(H, L(H, H))$ then $S^* \in L(H, L(H, H))$ is defined to be $S^*(h) = [S(h)]^*$, $h \in H$. The above equation can be written in the stochastic differential form, $s \leqslant t \leqslant \tau$,

$$d_s Y^*_{s,t} = -K_x(s, X_s)^*\, \Delta Y^*_{s,t}\, dW_s - \sigma_x(s, X_s, u(s))^* \circ Y^*_{s,t}\, ds. \tag{28}$$

Substitute (28) into (27) and note that $Y_{s,s} = I$, it follows that

$$d\phi_s = -f_x(s, X_s, u(s))\, ds - K_x(s, X_s)^*\, \phi_s\, dW_s - \sigma_x(s, X_s, u(s))^*\, \phi_s\, ds.$$

Hence ϕ_s satisfies Eq. (5). (Note that $A_x = K_x$.)

4. Example

Example 1. Consider the system

$$dX_t = dW_t - (CX_t + u(t)\, z_0)\, dt, \qquad 0 \leqslant t \leqslant \tau, \tag{29}$$

where $C \in L(B, H)$ and $z_0 \in H$. The control region is $[-1, 1]$ in the real line. The cost function is given by $f(t, x, u) = \langle Ax, z_0\rangle + u^2$ and $p \equiv 0$. In this case, the adjoint system is,

$$d\phi_t = (C^*\phi_t - A^*z_0)\, dt, \qquad \phi_\tau = 0.$$

The solution is easily seen to be

$$\phi_t = (\tau - t)\, e_*((t - \tau)\, C^*)\, A^*z_0\,, \tag{30}$$

where $e_*(S) = \sum_{n=0}^{\infty} S^n/(n+1)!$ for $S \in L(H, H)$.

Let $c_1(t) = \langle \phi_t\,, z_0\rangle$ and $c_2(t) = E\langle AX_t\,, z_0\rangle - E\langle \phi_t\,, CX_t\rangle$, then we have

$$E\mathscr{H}(\phi_t\,, X_t\,, u; t) = u^2 - c_1(t)u + c_2(t).$$

Obviously,

$$\inf_{-1\leqslant u\leqslant 1} E\mathscr{H}(\phi_t\,, X_t\,, u; t) = \begin{cases} -(c_1(t)^2/4) + c_2(t), & \text{if } \ |c_1(t)| \leqslant 2; \\ 1 - c_1(t) + c_2(t), & \text{if } \ c_1(t) > 2; \\ 1 + c_1(t) + c_2(t), & \text{if } \ c_1(t) < -2. \end{cases}$$

Therefore by Theorem 1, in order for $u(t)$ to be optimal it must be

$$u(t) = \begin{cases} c_1(t)/2, & \text{if } \ |c_1(t)| \leqslant 2; \\ \ \ 1, & \text{if } \ c_1(t) > 2; \\ -1, & \text{if } \ c_1(t) < -2. \end{cases}$$

The corresponding trajectory, i.e., the solution of (29), is given by

$$X_t = e^{-Ct}x + \int_0^t e^{C(s-t)}\, dW_s - \left[\int_0^t e^{C(s-t)}u(s)\, ds\right] z_0\,.$$

As a matter of fact, by putting $u(t)$ and X_t into Φ in (3) it can be checked that $u(\cdot)$ is the optimal control.

Example 2. Consider one dimensional case, i.e., $B = H = \mathbb{R}$. Let b_t be one dimensional Brownian motion starting at the origin. Consider the system with $x_0 = 0$,

$$dx_t = x_t\, db_t + \{(1/2)\, x_t + u(t)\}\, dt, \qquad 0 \leqslant t \leqslant \tau.$$

The control region U is $\mathbb{R}$ and the cost function is quadratic, i.e., $f(t, x, u) = x^2 + xu + u^2$, $p = 0$.

Take expectation in both sides of the system, it follows immediately that

$$Ex_t = \int_0^t e^{(t-s)/2} u(s)\, ds. \tag{31}$$

The adjoint system is

$$d\phi_t = -\phi_t\, db_t - [(1/2)\,\phi_t + 2x_t + u(t)]\, dt,$$

$$\phi_\tau = 0.$$

We have

$$E\phi_t = \int_t^\tau e^{(s-t)/2}[2Ex_s + u(s)]\, ds. \tag{32}$$

Now, the Hamiltonian is

$$\mathscr{H}(\phi_t, x_t, u; t) = u^2 + (x_t + \phi_t)u + x_t^2 + (1/2)\, x_t\phi_t .$$

Hence

$$\inf_{-\infty < u < \infty} E\mathscr{H}(\phi_t, x_t, u; t) = \beta(t) - \alpha(t)^2/4,$$

where $\alpha(t) = Ex_t + E\phi_t$ and $\beta(t) = Ex_t^2 + (1/2)\, Ex_t\phi_t$.
Therefore, by Theorem 1 in order for $u(\cdot)$ to be optimal we must have

$$u(t) = -\alpha(t)/2.$$

From the above equation, (31) and (32) we can derive a differential equation for $u(t)$, i.e.,

$$u''(t) = 3/4u(t)$$

with "initial" conditions

$$u(0) + 2u'(0) = 0 \qquad \text{and} \qquad 3u(\tau) + 2u'(\tau) = 0.$$

It is then easy to see that $u \equiv 0$. Obviously, this $u(\cdot)$ is the optimal control by noting that $\Phi \geqslant 0$.

5. Proof of Theorem 2

Lemma 6. $EX_t = \int_0^t [\exp(\int_s^t C(\lambda)\, d\lambda)]\, \theta(s, u(s))\, ds$, *where* $C(\lambda)$ *is regarded as an operator of* H.

Remark. Although X_t is a process in B, EX_t is a process in H.

Proof. From the following equation

$$X_t = \int_0^t A(s, X_s)\, dW_s + \int_0^t [C(s)\, X_s + \theta(s, u(s))]\, ds,$$

we have, since $E \int_0^t A(s, X_s)\, dW_s = 0$ (see [8]),

$$EX_t = \int_0^t [C(s)\, EX_s + \theta(s, u(s))]\, ds.$$

The lemma follows by solving the above H-valued integral equation.

Lemma 7. $E \mid DX_t \mid^2 = \int_0^t [\exp(2 \int_s^t \alpha(\lambda)\, d\lambda)]\, h(s)\, ds$, *where* $h(s) = 2\langle EX_s\, , D^*D\theta(s, u(s))\rangle + \operatorname{trace} D^*DJJ^*$.

Proof. Let $\phi(x) = \mid Dx \mid^2$. By Ito's formula [8],

$$d\phi(X_t) = (A^*(t, X_t)\, \phi'(X_t), dW_t) + \{\langle \phi'(X_t), C(t)\, X_t + \theta(t, u(t))\rangle + (1/2) \operatorname{trace}[A^*(t, X_t)\, \phi''(X_t)\, A(t, X_t)]\}\, dt. \tag{33}$$

But $\phi'(x) = 2D^*Dx$ and $\phi''(x) = 2D^*D$. Therefore,

$$\begin{aligned} \langle \phi'(X_t), C(t)\, X_t\rangle &= 2\langle D^*DX_t\, , C(t)\, X_t\rangle, \\ &= 2\langle DX_t\, , DC(t)\, X_t\rangle, \\ &= 2\langle DX_t\, , \alpha(t)\, DX_t\rangle, \\ &= 2\alpha(t) \mid DX_t \mid^2, \\ &= 2\alpha(t)\, \phi(X_t), \end{aligned}$$

and

$$(1/2) \operatorname{trace}[A^*(t, X_t)\, \phi''(X_t)\, A(t, X_t)] = \operatorname{trace} D^*DJJ^*.$$

Here we have used the assumption that $DC(t) = \alpha(t)D$ and $DK(t, x) = 0$. Note that $E \int_0^t (A^*(s, X_s)\, \phi'(X_s), dW_s) = 0$ (see [8]). Hence from Eq. (33) we have

$$E\phi(X_t) = 2 \int_0^t \alpha(s)\, E\phi(X_s)\, ds + \int_0^t \{2\langle EX_s\, , D^*D\theta(s, u(s))\rangle + \operatorname{trace} D^*DJJ^*\}\, ds.$$

The lemma then follows easily by solving the above ordinary integral equation.

Now, from Lemma 6 and Lemma 7 it follows that the functional Φ in (3), defined in the function space $\mathscr{U}$, is continuous. This can be checked as follows: If $u_n(\cdot)$ converges to $u(\cdot)$ in $\mathscr{U}$ then $u_n(\cdot)$ converges to $u(\cdot)$ almost everywhere. But every map is bounded and continuous, therefore Lebesgue dominated convergence theorem can be applied to conclude that $\Phi(u_n)$ converges to $\Phi(u)$.

Finally, we recall some facts about abstract Wiener space. $\mathscr{U}$ is a real separable Hilbert space and the operator T defined in (8) is a Hilbert–Schmidt operator of $\mathscr{U}$. Let $\hat{\mathscr{U}} = (\ker T)^{\perp}$. Then T is a one-to-one Hilbert–Schmidt operator of $\hat{\mathscr{U}}$. It is well-known that $(T(\hat{\mathscr{U}}), \hat{\mathscr{U}})$ is an abstract Wiener space and that the set $\{Tu; u \in \hat{\mathscr{U}}, \| u \|_0 \leqslant r\}$ is compact in $\hat{\mathscr{U}}$-topology. Clearly this set is also compact in $\mathscr{U}$. But this set is the same as $\{Tu; u \in \mathscr{U}, \| u \|_0 \leqslant r\} = \mathscr{U}_{T,r}$. Therefore, we conclude that $\mathscr{U}_{T,r}$ is compact in $\mathscr{U}$. Now, since Φ is continuous in $\mathscr{U}$, Φ assumes both maximum and minimum over $\mathscr{U}_{T,r}$.

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